Understanding Confidence Intervals

1 What is a confidence interval?

In practice a confidence interval is used to express the uncertainty in a quantity being estimated. There is uncertainty because inferences are based on a random sample of finite size from a population or process of interest. To judge the statistical procedure we can ask what would happen if we were to repeat the same study, over and over, getting different data (and thus different confidence intervals) each time.

2 Objectives

The lesson is designed to illustrate aspects of confidence interval computation and interpretation. By the end of the lesson students should:

- Know that a confidence interval computed from one sample will be different from a confidence interval computed from another sample.
- Understand the relationship between sample size and width of confidence interval.
- Understand the relationship between confidence level and width of confidence interval.
- Know that sometimes the computed confidence interval does not contain the true mean value (that is, it is incorrect) and understand how this coverage rate is related to confidence level.

3 Startup Instructions

The startup instructions are given using the confidence interval module as the example. Substitute the other module names to run another.
On a Unix workstation

```
% ci_module
```

On a PC

- Click on the Lisp-Stat icon in the program manager window
- Click on the `ci.lsp` icon in the Lisp-Stat window

On a Macintosh

- Start up xlispstat, by clicking on the XLispStat icon
- Pull down the `File` menu and select `Load`
- Select the folder `Teach`
- Select `ci.lsp`

4 The module interface

The uncertainty associated with confidence intervals is shown by dynamically resampling and displaying computed confidence intervals. The module that is shown in Figure 1 simulates taking samples from a normal population, with mean μ and standard deviation σ, and computing $100(1-\alpha)\%$ confidence intervals for the population mean, μ (assuming unknown variance):

$$(\bar{X} - t_{\alpha/2}\frac{s}{\sqrt{n}}, \bar{X} + t_{\alpha/2}\frac{s}{\sqrt{n}})$$

Four confidence intervals, computed from four different samples, are shown simultaneously in

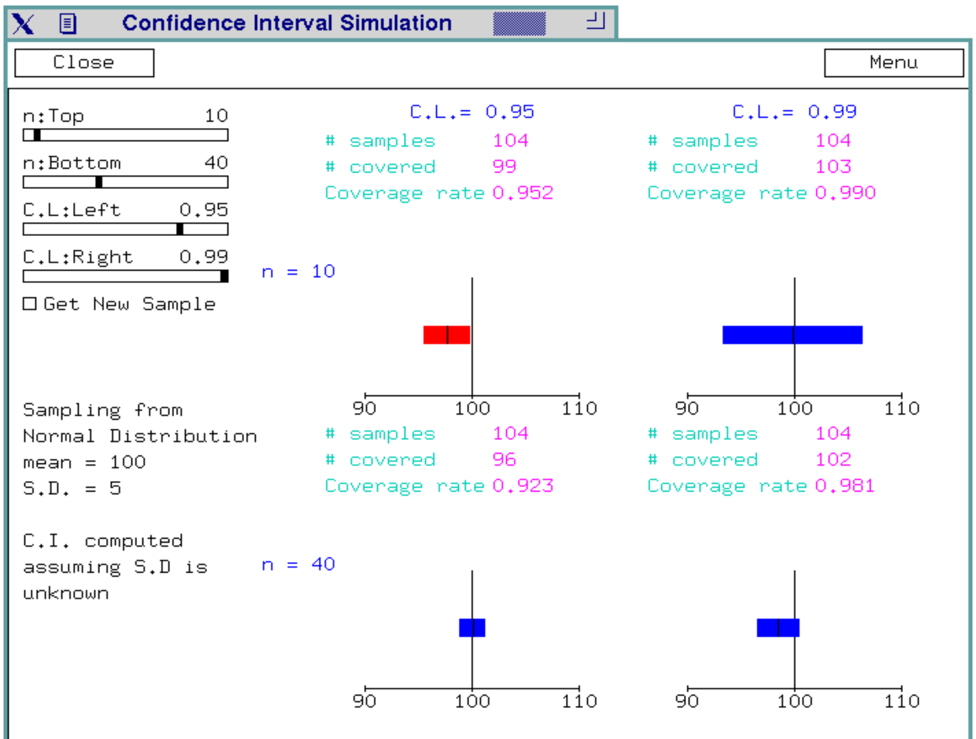

Figure 1: *Confidence Interval Teaching Module. 104 samples have been taken and the coverage rate for n=10, C.L.=0.95 is 0.952, for n=10 C.L.=0.99 is 0.990, for n=40, C.L.=0.95 is 0.923, for n=40, C.L.=0.99 is 0.981. The confidence interval for n=10, C.L.=0.95 does NOT cover the true mean.*

a 2×2 grid. The top row of the grid shows confidence intervals for samples of size 10 and the bottom row shows confidence intervals for samples of size 40. The left column shows 95% confidence intervals and the right column shows 99% confidence intervals. So the top left plot shows a 95% confidence interval on a sample of size 10. The top right plot shows a 99% confidence interval on a sample of size 10. The bottom left plot shows a 95% confidence interval on a sample of size 40, and the bottom right plot shows a 99% confidence interval on a sample of size 40. The true population mean is represented by the long fixed vertical black line.

5 Warmups

To gain some familiarity with the module, try the following:

1. Click on `Get New Sample` (or hold it down) and watch the changes in the computed confidence intervals as a new sample is taken.

2. Change the sample size for the intervals by sliding one of the sample size scrollbars, and compute some confidence intervals for a few different samples.

3. Change the confidence level by sliding one of the C.L. scrollbars, and compute some new confidence intervals for a few different samples.

4. Reset the simulation by shifting one of the scrollbars forward and back again.

6 Exercises

1. Look closely at the start up window of the confidence interval module. There are two features to notice.

   (a) The confidence intervals for sample size n=10 are larger than the confidence intervals for sample size n=40. Why should this be so?

   (b) The confidence intervals with confidence level 99% are larger than the confidence intervals with confidence level 95%. Why should this be?

2. Take a new sample and examine the new computed confidence intervals. Why are the intervals different than those calculated from the first samples?

3. Continue sampling until any interval shown is RED. How is correctness of a confidence interval defined? What do you notice about the true mean value (vertical black bar) and the position of the confidence interval? Describe what you see in terms of correctness.

4. Reset the simulation by shifting one of the scrollbars forward and back again. The number of samples should now be reset to 1.

5. Take 10 samples. Write the value of `Coverage Rate` for the different sample sizes and confidence levels in Table 1.

6. Continue getting new samples and record the coverage rates for 20, 50, 100 and 500 samples in Table 1. (You can simply hold the button down rather than do individual clicks.)

7. Comment on how the coverage rate compares to the confidence level as the number of samples taken increases (for each `n` and `C.L.`).

8. Explain the real meaning behind the term "95% confident."

Coverage Rate	10 samples	20 samples	50 samples	100 samples	500 samples
$n = 10,\ CL = 0.95$					
$n = 10,\ CL = 0.99$					
$n = 40,\ CL = 0.95$					
$n = 40,\ CL = 0.99$					

Table 1: Simulation results

7 Solutions to Exercises

5.1.a.i The width of a confidence interval is inversely proportional to $\sqrt{n}$ so as n increases the width decreases. Intuitively the reasoning is that as sample size increases the variation in sample mean observed from one sample to the next decreases.

5.1.a.ii To be more confidenct (99% opposed to 95%) that the confidence interval contains the true mean the interval needs to be wider

5.1.b Each sample is different. Hence the sample mean and sample standard deviation, which are used to compute a confidence interval, differ from one sample to the next.

5.1.c An interval is correct if it contains the true parameter value. (Of course, there is no way of knowing if an interval is correct in practice.) The RED confidence interval does not contain the true mean value so the interval is not correct.

5.2.d As the number of samples taken increases, the coverage rate should get closer to the confidence level.

5.2.e The term “95% confident” means that there is a 95% chance that any one interval will contain the true parameter value, given that the standard assumptions are true.